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# The Best Strong Uniqueness Constant for a Multivariate Chebyshev Polynomial

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## INTRODUCTION

Let $B$ be a compact set in $\mathbb{R}^r$, $r \geqslant 1$. Suppose $f \in C(B)$ and $V$ to be a finite-dimensional subset of $C(B)$. $v_0 \in V$ is called a best approximation on $B$ to $f$ out of $V$ if $\|f - v_0\| \leqslant \|f - v\|$ for all $v \in V$. A best approximation to a given $f$ always exists, and it is sometimes unique. Newman and Shapiro [5] defined a quantitative notion of uniqueness (which implies uniqueness) called strong uniqueness. Namely, $v_0$ is a strongly unique best approximation to $f$ if there exists a *positive* constant, $\gamma(f, B, V)$ such that

$$\|f - v\| \geqslant \|f - v_0\| + \gamma \|v - v_0\| \tag{1}$$

holds for all $v \in V$. The largest $\gamma$ such that (1) holds is called the *best strong uniqueness constant for $f$*. While many aspects of best strong uniqueness constants have been studied, particularly in the case $r = 1$ (cf. Henry and Swetits [4]) the best strong uniqueness constant is known explicitly only (essentially) for the univariate Chebyshev polynomial.

The present work is devoted to obtaining the best strong uniqueness constant for a multivariate Chebyshev polynomial. This will be accomplished by using the following characterization of the best strong uniqueness constant (implicit in Bartelt and McLaughlin [1]). For $g \in C(B)$ let $E(g; B) = \{x \in B : |g(x)| = \|g\|\}$.

Given $f \in C(B)$ and a $v_0 \in V$, if

$$\min_{\substack{v \in V \\ \|v\| = 1}} \max_{x \in E(f - v_0; B)} \operatorname{Re}[\operatorname{sgn} \overline{f(x) - v_0(x)}]\, v(x) = \gamma^* > 0, \tag{2}$$

then $v_0$ is a strongly unique best approximation to $f$ on $B$ out of $V$ and $\gamma^*$ is the best strong uniqueness constant for $f$. The converse also holds.



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In the first section we collect some basic information about univariate Chebyshev polynomials, establish our notation for multivariate polynomials, and define the multivariate Chebyshev polynomial we shall be considering. Then we state our result and sketch the proof. The second section is devoted to the details of the proof.

## 1. Statement of Result

The univariate Chebyshev polynomial of degree $k$ ($k = 0, 1, 2, \ldots$) is defined by $T_k(u) = \cos k\theta$, where $u = \cos\theta, 0 \leqslant \theta \leqslant \pi$. Clearly, $|T_k(u)| \leqslant 1$, $-1 \leqslant u \leqslant 1$, with equality holding only if

$$u = \eta_j^{(k)} := \cos\frac{j\pi}{k}, \qquad j = 0, 1, \ldots, k.$$

In fact,

$$T_k(\eta_j^{(k)}) = (-1)^j, \qquad j = 0, 1, \ldots, k. \tag{3}$$

Furthermore, 0 is a strongly unique best approximation to $T_k$ $(k \geqslant 1)$ on $[-1, 1]$ out of the polynomials of degree at most $k - 1$, and the best strong uniqueness constant for $T_k$ is $(2k - 1)^{-1}$ (cf. Cline [2]).

It is also important to recall that the Chebyshev polynomials are orthogonal on the points $\eta_0^{(k)}, \ldots, \eta_k^{(k)}$. This orthogonality property, in the cases that will interest us here, is given by

$$\sum_{s=0}^{k}{}'' T_a(\eta_s^{(k)})\, T_b(\eta_s^{(k)}) = \begin{cases} k & \begin{cases} a = b = 0, \text{ or} \\ a = b = k, \text{ or} \\ (a, b) = (0, 2k). \end{cases} \\ \dfrac{k}{2} & \begin{cases} b = 2k - a \text{ and both} \\ a \neq b \text{ and } (a, b) \neq (0, 2k) \\ \text{or} \\ a = b \text{ and } a \neq 0, k. \end{cases} \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $a = 0, 1, \ldots, k, b = 0, 1, \ldots, 2k$, and the two strokes on the summation sign mean that the first and last summands are to be halved (cf. Rivlin [7]).

We turn now to multivariate polynomials. Let $\mathbf{x}$: $(x_1, \ldots, x_r)$ be a point in $\mathbb{R}^r$, $r \geqslant 1$. For each index $\mathbf{k}$: $(k_1, \ldots, k_r)$, the $k_i$ being non-negative integers, we put $|\mathbf{k}| = k_1 + \cdots + k_r$. Suppose we are given $\mathbf{n}: (n_1, \ldots, n_r), \mathbf{n} \neq \mathbf{0}$. Let $V$ be the real linear space spanned by $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_r^{i_r}$, where $0 \leqslant i_j \leqslant n_j, j = 1, \ldots, r$ and $\mathbf{i} \neq \mathbf{n}$. That is, $V$ consists of all polynomials

$$v(\mathbf{x}) = \sum_{\substack{\mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n} \\ \mathbf{i} \neq \mathbf{n}}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}.$$

Note that the dimension of $V$ is $(n_1 + 1) \cdots (n_r + 1) - 1$, and that if we put

$$T_{\mathbf{k}}(\mathbf{x}) = T_{k_1}(x_1) \cdots T_{k_r}(x_r),$$

then $\{T_{\mathbf{i}}(\mathbf{x})\}$ with $\mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n}$ and $\mathbf{i} \neq \mathbf{n}$ also forms a basis for $V$. It is known (Ehlich and Zeller [3] and Reimer [6]) that $v_0 = 0$ is the unique best approximation on $I^r := [-1, 1]^r$ to $T_{\mathbf{n}}(\mathbf{x})$ out of $V$ (a fact which follows from our result). We shall show that the best strong uniqueness constant for $T_{\mathbf{n}}(\mathbf{x})$ is

$$\gamma^*(T_n, I^r, V) = \left(2^r \prod_{i=1}^{r} n_i - 1\right)^{-1}. \tag{5}$$

In case $r = 1$ we thus recover the previously mentioned result of Cline.

The proof uses (2), with $f = T_{\mathbf{n}}$, $v_0 = 0$ and $B = I^r$. The set $E(T_{\mathbf{n}}; I^r)$ consist of all points $\boldsymbol{\eta}_{\mathbf{j}}: (\eta_{j_1}^{(n_1)}, \ldots, \eta_{j_r}^{(n_r)})$ with $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}$ and so we are required to show (in view of (3)) that

$$\min_{\substack{v \in V \\ \|v\| = 1}} \max_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}} T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}})\, v(\boldsymbol{\eta}_{\mathbf{j}}) = \left(2^r \prod_{i=1}^{r} n_i - 1\right)^{-1} =: \lambda. \tag{6}$$

This we propose to accomplish in two parts. First we shall show that the maximum in (6) cannot be less than $\lambda$ for any $v$ satisfying $\|v\| = 1$. Then we exhibit a $v$, with $\|v\| = 1$, such that the maximum in (6) does not exceed $\lambda$.

## 2. The Proof of (5)

It suffices to establish (6).

(i) Suppose that

$$T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}})\, v(\boldsymbol{\eta}_{\mathbf{j}}) < \lambda, \qquad \text{all } \mathbf{j}, \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}, \tag{7}$$

for some $v \in V$ satisfying $\|v\| = 1$. Then for $p(\mathbf{x}) = \lambda - T_{\mathbf{n}}(\mathbf{x})\, v(\mathbf{x})$ we have

$$p(\boldsymbol{\eta}_{\mathbf{j}}) > 0, \qquad \text{all } \mathbf{j}, \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}. \tag{8}$$

Now, if $D = \{\mathbf{i}: \mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n}, \mathbf{i} \neq \mathbf{n}\}$, then we may write

$$v(\mathbf{x}) = \sum_{\mathbf{i} \in D} A_{\mathbf{i}} T_{\mathbf{i}}(\mathbf{x}) \tag{9}$$

and

$$T_{\mathbf{n}}(\mathbf{x})\, v(\mathbf{x}) = \sum_{\mathbf{i} \in D} A_{\mathbf{i}} T_{\mathbf{i}}(\mathbf{x})\, T_{\mathbf{n}}(\mathbf{x}). \tag{10}$$

If we recall the identity $T_a(u)\, T_b(u) = \frac{1}{2}(T_{a+b}(u) + T_{|b-a|}(u))$, then for $\mathbf{i} \in D$

$$\begin{aligned} T_{\mathbf{i}}(\mathbf{x})\, T_{\mathbf{n}}(\mathbf{x}) &= 2^{-r} \prod_{s=1}^{r} (T_{n_s + i_s}(x_s) + T_{n_s - i_s}(x_s)) \\ &= 2^{-r} \sum_{\boldsymbol{\mu} \in \Delta(\mathbf{i})} c_{\boldsymbol{\mu}} T_{\boldsymbol{\mu}}(\mathbf{x}), \end{aligned}$$

where the set of indices $\Delta(\mathbf{i})$ and the multiplicities $c_{\boldsymbol{\mu}}$ have the following properties: $\Delta(\mathbf{i}) = \{((n_1 \pm i_1),..., (n_r \pm i_r))\}$ where all possible sign sequences are taken on. Thus if exactly $l$ ($l = 0,..., r$) of the components of $\mathbf{i}$ are zero, then $\Delta(\mathbf{i})$ consists of $2^{r-l}$ distinct indices and $c_{\boldsymbol{\mu}} = 2^l$ for $\boldsymbol{\mu} \in \Delta(\mathbf{i})$.

Hence if

$$T_{\mathbf{n}}(\mathbf{x})\, v(\mathbf{x}) = \sum_{\boldsymbol{\mu} \in \Delta} B_{\boldsymbol{\mu}} T_{\boldsymbol{\mu}}(\mathbf{x}), \tag{11}$$

then

$$\Delta = \bigcup_{\mathbf{i} \in D} \Delta(\mathbf{i})$$

and in view of (10), if $\boldsymbol{\mu} \in \Delta(\mathbf{i})$ and exactly $l$ ($l = 0, 1,..., r$) components of $\mathbf{i}$ are zero

$$B_{\boldsymbol{\mu}} = \frac{A_{\mathbf{i}}}{2^{r-l}}. \tag{12}$$

Thus

$$p(\mathbf{x}) = \sum_{\boldsymbol{\mu} \in \Delta_0} \beta_{\boldsymbol{\mu}} T_{\boldsymbol{\mu}}(\mathbf{x})$$

where $\Delta_0 = \Delta \cup \{\mathbf{0}\}$, $\beta_{\mathbf{0}} = \lambda$ and $\beta_{\boldsymbol{\mu}} = -B_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \neq \mathbf{0}$.

We are next going to bound the $\beta_\mu$, hence the $A_i$, by utilizing the orthogonality property (4), and the bounds obtained will contradict the hypothesis that $\|v\| = 1$. Suppose that $\mathbf{0} \leqslant \mathbf{v} \leqslant \mathbf{n}$, then

$$T_\mathbf{v}(\boldsymbol{\eta}_\mathbf{j})\, p(\boldsymbol{\eta}_\mathbf{j}) = \sum_{\boldsymbol{\mu} \in \Delta_0}' \beta_\mu T_\mathbf{v}(\boldsymbol{\eta}_\mathbf{j})\, T_\mu(\boldsymbol{\eta}_\mathbf{j})$$

for all $\mathbf{j}$ satisfying $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}$. Thus

$$\begin{aligned}\sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_\mathbf{v}(\boldsymbol{\eta}_\mathbf{j})\, p(\boldsymbol{\eta}_\mathbf{j}) &:= \sum_{j_1=0}^{n_1}{}'' \cdots \sum_{j_r=0}^{n_r}{}'' \prod_{m=1}^{r} T_{v_m}(\eta_{j_m}^{(n_m)})\, p(\boldsymbol{\eta}_j) \\ &= \sum_{\boldsymbol{\mu} \in \Delta_0}' \beta_\mu \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_\mathbf{v}(\boldsymbol{\eta}_\mathbf{j})\, T_\mu(\boldsymbol{\eta}_\mathbf{j}) \\ &= \sum_{\boldsymbol{\mu} \in \Delta_0}' \beta_\mu \sum_{j_1=0}^{n_1}{}'' \cdots \sum_{j_r=0}^{n_r}{}'' \prod_{m=1}^{r} T_{v_m}(\eta_{j_m}^{(n_m)})\, T_{\mu_m}(\eta_{j_m}^{(n_m)}). \qquad (13)\end{aligned}$$

If we now apply (4) consecutively with $a = v_m$, $b = \mu_m$, $s = j_m$ and $k = n_m$, $m = 1,\ldots, r$ to the expression at the end of the chain of equalities in (13), and think of $\mathbf{v}(\mathbf{0} \leqslant \mathbf{v} \leqslant \mathbf{n})$ as fixed, we obtain the following results:

Let $R = \{1,\ldots, r\}$ and $R_k$ be the set of all distinct $k$-tuples of integers in $R$, $k = 0,\ldots,r$ $(R_0 = \varnothing)$. Suppose $\mathbf{0} \leqslant \mathbf{v} \leqslant \mathbf{n}, \mathbf{0} \neq \mathbf{v}$ and for $k \in \{0, 1,\ldots, r\}$ and some fixed $r_k \in R_k$

$$v_m = 0, n_m\,;\, m \in r_k, \qquad 0 < v_m < n_m\,;\, m \in R \backslash r_k. \qquad (14)$$

Then, in view of the definition of $p$ and (8) we have

$$\begin{aligned}\frac{\prod_{m=1}^{r} n_m}{2^{r-k}} \left| \sum_{\boldsymbol{\mu} \in \Delta(\mathbf{n}-\mathbf{v})}' \beta_\mu \right| &\leqslant \left| \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_\mathbf{v}(\boldsymbol{\eta}_\mathbf{j})\, p(\boldsymbol{\eta}_\mathbf{j}) \right| \\ &\leqslant \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' p(\boldsymbol{\eta}_\mathbf{j}) = \lambda \prod_{m=1}^{r} n_m - \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_\mathbf{n}(\boldsymbol{\eta}_\mathbf{j})\, v(\boldsymbol{\eta}_\mathbf{j}) = \lambda \prod_{m=1}^{r} n_m. \qquad (15)\end{aligned}$$

(The fact that

$$\sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_\mathbf{n}(\boldsymbol{\eta}_\mathbf{j})\, v(\boldsymbol{\eta}_\mathbf{j}) = 0$$

follows from (4) since $\mathbf{i} \neq \mathbf{n}$ in (10).) Now for $\mathbf{v} \neq \mathbf{0}$ satisfying (14) and $\boldsymbol{\mu} \in \Delta(\mathbf{n} - \mathbf{v})$

$$-\beta_\mu = B_\mu = \frac{A_{\mathbf{n}-\mathbf{v}}}{2^{r-l}}$$

and $\Delta(\mathbf{n}-\mathbf{v})$ consists of $2^{r-l}$ indices. Thus

$$\left| \sum_{\mu \in \Delta(\mathbf{n}-\mathbf{v})} \beta_{\mu} \right| = |A_{\mathbf{n}-\mathbf{v}}|,$$

and we conclude from (15) that

$$|A_{\mathbf{n}-\mathbf{v}}| \leqslant 2^{r-k} \lambda \tag{16}$$

for all $\mathbf{v} \neq \mathbf{0}$ satisfying (14). For each $k = 0, 1, \ldots, r$, (16) provides an upper bound on

$$2^k \sum_{r_k \in R_k} \prod_{m \in R \setminus r_k} (n_m - 1) \tag{17}$$

distinct coefficients of $v$, except that when $k = r$ we must subtract 1 (obtaining $2^r - 1$) since $\mathbf{v} \neq \mathbf{0}$ (or $\mathbf{i} \neq \mathbf{n}$).

Note that when $k = r$ and $\mathbf{v} = \mathbf{n}$

$$\left| \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' T_{\mathbf{v}}(\boldsymbol{\eta}_{\mathbf{j}}) p(\boldsymbol{\eta}_{\mathbf{j}}) \right| = \left| \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' (-1)^{|\mathbf{j}|} p(\boldsymbol{\eta}_{\mathbf{j}}) \right| < \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}}'' p(\boldsymbol{\eta}_{\mathbf{j}})$$

since $|\mathbf{j}|$ cannot be even (or odd) for all $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}$, and (15) informs us that strict inequality holds in (16), i.e.,

$$|A_{\mathbf{0}}| < \lambda. \tag{18}$$

If we also observe that $\{\mathbf{v}\colon \mathbf{0} \leqslant \mathbf{v} \leqslant \mathbf{n}, \mathbf{v} \neq \mathbf{0}\}$ is in one-to-one correspondence with $\{\mathbf{i} : \mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n}, \mathbf{i} \neq \mathbf{n}\} = D$, then (9), (16), (17), and (18) imply that

$$\|v\| < \left( \left( \sum_{k=0}^{r} 2^r \sum_{r_k \in R_k} \prod_{m \in R \setminus r_k} (n_m - 1 \right) - 1 \right) \lambda =: C_r \lambda. \tag{19}$$

It is easy to evaluate $C_r$. Namely, let $y_m = n_m - 1$, $m = 1, \ldots, r$, and consider the polynomial

$$t(u) = (u - y_1) \cdots (u - y_m) = \sum_{m=0}^{r} (-1)^m \sigma_m u^{r-m},$$

where $\sigma_0 = 1$ and $\sigma_1, \ldots, \sigma_m$ are the elementary symmetric functions of $y_1, \ldots, y_r$. Then

$$C_r = 2^r(-1)^r t(-1) - 1 = 2^r \prod_{m=1}^{r} n_m - 1 = \lambda^{-1}. \tag{20}$$

Equations (20) and (19) yield $\|v\| < 1$, a contradiction. Inequality (7) cannot hold for any $v \in V$ with $\|v\| = 1$ and so

$$\max_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}} T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}})\, v(\boldsymbol{\eta}_{\mathbf{j}}) \geqslant \lambda, \qquad \text{all } v \in V, \|v\| = 1. \tag{21}$$

(ii) We shall prove that (6) holds by exhibiting a $v$ of norm one for which equality holds in (21). Namely,

$$\begin{aligned} w(x) &= \left(2^r \sum_{i_1=0}^{n_1}{}'' \cdots \sum_{i_r=0}^{n_r}{}'' T_{i_1}(x_1) \cdots T_{i_r}(x_r)\right) - T_{\mathbf{n}}(\mathbf{x}) \\ &= \left(2^r \sum_{\mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n}}{}'' T_{\mathbf{i}}(\mathbf{x})\right) - T_{\mathbf{n}}(\mathbf{x}) \end{aligned}$$

clearly satisfies $w \in V$, $\|w\| = \lambda^{-1}$.

But if we put

$$q_k(u) = \sum_{s=0}^{k}{}'' T_s(u),$$

then for $a = 0, 1, \ldots, k$

$$q_k(\eta_a^{(k)}) = \sum_{s=0}^{k}{}'' T_s(\eta_a^{(k)}) = \sum_{s=0}^{k}{}'' T_a(\eta_s^{(k)}) = \begin{cases} k, & a = 0, \\ 0, & \text{otherwise}, \end{cases}$$

in view of (4). Thus, for $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{n}$,

$$w(\boldsymbol{\eta}_{\mathbf{j}}) = 2^r q_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}}) - T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}}) = \begin{cases} \lambda^{-1}, & j = 0 \\ -T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}}), & \text{otherwise} \end{cases}$$

and

$$T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}})\, w(\boldsymbol{\eta}_{\mathbf{j}}) = \begin{cases} \lambda^{-1}, & j = 0 \\ -1, & \text{otherwise}. \end{cases}$$

Thus $v^* = -\lambda w$ satisfies $v^* \in V$, $\|v^*\| = 1$ and

$$\max_{\mathbf{0} \leqslant j \leqslant \mathbf{n}} T_{\mathbf{n}}(\boldsymbol{\eta}_{\mathbf{j}})\, v^*(\boldsymbol{\eta}_{\mathbf{j}}) = \lambda.$$

This concludes the proof.

## REFERENCES


1. M. W. BARTELT AND H. W. MCLAUGHLIN, Characterizations of strong unicity in approximation theory, *J. Approx. Theory* **9** (1973), 255–266.
2. A. K. CLINE, Lipschitz conditions on uniform approximation operators, *J. Approx. Theory* **8** (1973), 160–172.
3. H. EHLICH AND K. ZELLER, Čebyšev — Polynome in mehreren Veränderlichen, *Math. Z.* **93** (1966), 142–143.
4. M. S. HENRY AND J. SWETITS, On extremal sets and strong unicity constants for certain $C^{\infty}$ functions, *J. Approx. Theory* (1983).
5. D. J. NEWMAN AND H. S. SHAPIRO, Some theorems on Čebyšev approximation, *Duke Math. J.*, **30** (1963), 673–681.
6. M. REIMER, On multivariate polynomials of least deviation from zero on the unit cube, *J. Approx. Theory* **23** (1978), 65–69.
7. T. J. RIVLIN, "The Chebyshev Polynomials," Wiley, New York, 1974.